On pentagon, ten-term, and tetrahedron relations

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Abstract

The tetrahedron equation in a special substitution is reduced to a pair
of pentagon and one ten-term equations. Various examples of solutions
are found. *O*-doubles of Novikov, which generalize the Heisenberg double
of a Hopf algebra, provide a particular algebraic solution to the problem.

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1 Introduction

The Yang-Baxter equation (YBE) [25, 3] can be considered as a tool for both constructing and solving integrable two-dimensional models of statistical mechanics and quantum field theory [2, 9]. Recent progress in understanding of the algebraic structure, lying behind the YBE, has led to the theory of quasi-triangular Hopf algebras [7].

The tetrahedron (or three-simplex) equation (TE) [26] has been introduced as a three-dimensional generalization of the YBE. Before describing it in an abstract algebraic form, first consider an associative unital algebra A, and define an important notation to be used throughout the paper. Namely, for each set of integers $\{i_1, i_2, \ldots, i_m\} \subset \{1, 2, \ldots, n\}$, for $m < n$, define a mapping

$$\tau_{i_1 i_2 \ldots i_m} \colon A^{\otimes m} \to A^{\otimes n}$$

in such a way that

$$a \otimes b \otimes \ldots \otimes c \mapsto 1 \otimes \ldots \otimes a \otimes \ldots \otimes b \otimes \ldots \otimes c \otimes \ldots \otimes 1,$$

where $a, b, \ldots, c$ in the r.h.s. stand on $i_1, i_2, \ldots, i_m$ -th positions, respectively, and unit elements, on others. The notation to be used is as follows:

$$u_{i_1 i_2 \ldots i_m} = \tau_{i_1 i_2 \ldots i_m}(u), \quad u \in A^{\otimes m}, \tag{1.1}$$

which means that the subscripts indicate the way an element of the algebra $A^{\otimes m}$ is interpreted as an element of the algebra $A^{\otimes n}$.

Next, we shall find it convenient to use the "permutation operator" P, which is an (additional) element in $A^{\otimes 2}$, defined by

$$P a \otimes b = b \otimes a P, \quad P^2 = 1 \otimes 1, \quad a, b \in A. \tag{1.2}$$

The (constant) TE can be written as a nonlinear relation in $A^{\otimes 6}$ on an invertible element $R \in A^{\otimes 3}$ as follows:

$$R_{123} R_{145} R_{246} R_{356} = R_{356} R_{246} R_{145} R_{123}, \tag{1.3}$$

where we have used the notation (1.1). One can introduce also the higher-dimensional analogs of the YBE [6], the n-simplex equations. For example, the (constant) four-simplex equation (FSE) is a relation in $A^{\otimes 10}$ on an invertible element $B \in A^{\otimes 4}$:

$$B_{0123} B_{0456} B_{1478} B_{2579} B_{3689} = B_{3689} B_{2579} B_{1478} B_{0456} B_{0123}. \tag{1.4}$$

Many solutions have been found already for the TE, e.g. [26, 4, 5, 14, 15, 16, 10, 19], though, an adequate algebraic framework (an analog of quasi-triangular Hopf algebras) is still missing. Practically nothing is known for the higher-simplex equations.

The purpose of this paper is to make a step towards the algebraic theory of the TE. Our main result is that one and the same system of equations

$$S_{12} S_{13} S_{23} = S_{23} S_{12}, \tag{1.5}$$

$$\overline{S}_{23}\overline{S}_{13}\overline{S}_{12} = \overline{S}_{12}\overline{S}_{23}, \tag{1.6}$$

in $A^{\otimes 3}$ and

$$\overline{S}_{12}S_{13}\overline{S}_{14}S_{24}\overline{S}_{34} = S_{24}\overline{S}_{34}S_{14}\overline{S}_{12}S_{13}, \tag{1.7}$$

in $A^{\otimes 4}$ on elements $S, \overline{S} \in A^{\otimes 2}$ implies both the TE for the combination

$$R^S_{123} = \overline{S}_{13}P_{23}S_{13}, \tag{1.8}$$

and the FSE for the combination

$$B^S_{0123} = \overline{S}_{13}P_{01}P_{23}S_{13}. \tag{1.9}$$

We should warn, however, that formula (1.9) is too restrictive to give genuinely four-dimensional models. As a matter of fact, it corresponds to non-interacting system of three-dimensional models.

The manifest symmetry properties of equations (1.5) – (1.7) are given by the following transformations of S and $\overline{S}$:

$$S_{12} \leftrightarrow \overline{S}_{21}, \quad S_{12} \leftrightarrow (\overline{S}_{12})^{-1}. \tag{1.10}$$

Equations (1.5) and (1.6) are the two forms of the celebrated pentagon equation (PE), which appears in various forms in representation theory of quantum groups as the Bidenharn-Elliot identity for the 6j-symbols, in quantum conformal field theory as an identity for the fusion matrices [20], in quasi-Hopf algebras as the consistency equation for the associator [8]. In the form of (1.5), (1.6) the PE first appeared in the geometric approach to three-dimensional integrable systems [17, 18]. In [18] a reduction of the TE to the PE has been suggested. Finally, the PE in the form (1.5), (1.6) was shown in [12] to be intimately related with the Heisenberg double of a Hopf algebra [22, 1, 23]. In particular, using the inclusion of the Drinfeld double into the tensor product of two Heisenbergs, one can reduce the YBE to the PE.

Equation (1.7), the "ten-term" relation, on two different solutions of the PE, appears to be satisfied by canonical elements in the O-double of a Hopf algebra, introduced in [21] as a generalization of the Heisenberg double. Thus, the O-double is, probably, the simplest algebraic framework for the TE.

The paper is organized as follows. In Sec. 2 particular solutions for the PE, which generalize the solutions, associated with the Heisenberg doubles of group algebras, are considered. The results of this section are used in the next one for construction of particular solutions (typically infinite dimensional) for the system (1.5) – (1.7). In Sec. 3 the latter system is derived from the TE, and FSE, and the examples of solutions are described. In Sec. 4 the O-double construction of a special class of solutions is presented.

2 Pentagon relation for a rational transformation

As is shown in [12], for a group G the operator

$$(S\varphi)(x,y) = \varphi(xy,y), \quad \varphi \in \mathcal{F}(G \times G), \quad x,y \in G, \tag{2.1}$$

in the space of functions on $G \times G$ satisfies PE (1.5). Actually, this is the "coordinate" representation for the canonical element in the Heisenberg double of the group algebra. In this section we generalize this result. Namely, let M be some set. Define an operator S in the space $\mathcal{F}(M \times M)$:

$$(S\varphi)(x, y) = \varphi(x \cdot y, x * y), \quad \varphi \in \mathcal{F}(M \times M), \qquad (2.2)$$

for some mappings

$$M \times M \xrightarrow{\cdot} M, \quad M \times M \xrightarrow{*} M. \qquad (2.3)$$

Let us call these the dot- and star-mapping, respectively. Imposing now PE (1.5) on S, we obtain the following equations:

$$(x \cdot y) \cdot z = x \cdot (y \cdot z), \qquad (2.4)$$

$$(x * y) \cdot ((x \cdot y) * z) = x * (y \cdot z), \qquad (2.5)$$

$$(x * y) * ((x \cdot y) * z) = y * z. \qquad (2.6)$$

If M is a group with respect to the dot-mapping, then equation (2.5) implies that

$$x * y = [x]^{-1} \cdot [x \cdot y], \qquad (2.7)$$

where an invertible mapping $[\ ] \colon M \to M$ is given by

$$[x] = 1 * x.$$

Substituting (2.7) into (2.6), we get the following functional equation

$$[[x]^{-1} \cdot [x \cdot y]] = \lambda(x) \cdot [y], \qquad (2.8)$$

with some function $\lambda \colon M \to M$. Putting $y = 1$ in (2.8), we obtain $\lambda(x) = 1$, and therefore $[x] = x$. Thus, we have proved the following proposition.

Proposition 1 *Let the set M be a group with respect to the dot-mapping. Then, the star-mapping of the form $x * y = y$ is the only solution to the system (2.5)–(2.6).*

Remind, that the corresponding S-operator, given by (2.1), is associated with the Heisenberg double of the group algebra. There are, however, other solutions, if the mappings under consideration are defined only for elements in "general position", e.g. birational mappings.

Proposition 2 *Let the set M be a subset of an associative ring with unit, and let the definitions*

$$x \cdot y = xy, \quad x * y = (1 - x^\epsilon)^{-\epsilon}(1 - (xy)^\epsilon)^\epsilon, \quad \epsilon = \pm 1, \qquad (2.9)$$

make sense in M. Then, equations (2.4) – (2.6) are satisfied.

The proof is straightforward.

Proposition 3 *Let $M = (0,1) \subset \mathbb{R}$ be the open unit interval with the dot-mapping given by the multiplication in $\mathbb{R}$, and let the star-mapping be continuously differentiable. Then, the system (2.5)–(2.6) is satisfied iff*

$$x * y = y \left(\frac{1 - x^{1/\alpha}}{1 - (xy)^{1/\alpha}} \right)^{\alpha}, \qquad (2.10)$$

where real $\alpha \geq 0$, and the case $\alpha = 0$ is understood as a limit $\alpha \to 0^+$.

Proof. It is straightforward to check that formula (2.10) solves the system (2.5)–(2.6. Let us prove that it is the only continuously differentiable solution.

Formulae (2.7) and (2.8) are still valid, with the mapping

$$[\,]\colon M \to \mathbb{R}_+$$

(rather than $M \to M$) being strictly increasing and continuously differentiable. Let us differentiate (2.8) with respect to y. The result can be written as

$$w\left([xy]/[x]\right) = w(y)/w(xy), \qquad (2.11)$$

where

$$w(x) = d \log x / d \log[x], \qquad (2.12)$$

while differentiation of (2.8) with respect to x together with (2.11) gives

$$w(xy) = w(x) - w(y)w(x)d \log \lambda(x)/d \log x. \qquad (2.13)$$

Consistency of the last equation under the permutation $x \leftrightarrow y$ fixes the derivative of the λ-function up to a real constant c:

$$d \log \lambda(x)/d \log x = c - 1/w(x).$$

Plugging this back into (2.13), we obtain the closed functional equation

$$w(xy) = w(x) + w(y) - cw(x)w(y). \qquad (2.14)$$

Here $c \neq 0$ (otherwise $w(x) \sim \log x$, and eqs. (2.11) and (2.12) are in contradiction), therefore, equation (2.14) can be rewritten in the form

$$1 - cw(xy) = (1 - cw(x))(1 - cw(y)),$$

the general continuous solution of which is well known:

$$1 - cw(x) = x^{1/\alpha}, \quad 1/\alpha \in \mathbb{R}.$$

Compatibility of eqs. (2.11) and (2.12) fixes $c = 1$. Thus,

$$w(x) = 1 - x^{1/\alpha} > 0 \Longrightarrow \alpha > 0,$$

where the first inequality follows from the definition (2.12) of $w(x)$ and the fact that $[x]$ is strictly monotonically increasing function. Finally, solving the differential equation (2.12), we complete the proof.

In conclusion, note that any two non-zero parameters $\alpha, \alpha' \neq 0$ in (2.10) give equivalent S-operators (2.2), consequently, we have only two inequivalent solutions for $M = (0,1)$, corresponding to $\alpha = 0$, and $\alpha = 1$.

3 Pentagon, ten-term, three-, and four-simplex relations

Consider the following "ansatz" for a solution to equation (1.3):

$$R_{123}^T = \overline{T}_{13}P_{23}T_{13}, \tag{3.1}$$

for invertible elments T, $\overline{T} \in A^{\otimes 2}$, with P being defined in (1.2).

Proposition 4 *The TE (1.3) for an element R of the form (3.1) is equivalent to the existence of invertible elements S, $\overline{S} \in A^{\otimes 2}$ such that the following equations are satisfied:*

$$S_{12}T_{13}T_{23} = T_{23}T_{12}, \quad \overline{T}_{23}\overline{T}_{13}\overline{S}_{12} = \overline{T}_{12}\overline{T}_{23}, \tag{3.2}$$

$$\overline{S}_{12}T_{13}\overline{T}_{14}T_{24}\overline{T}_{34} = T_{24}\overline{T}_{34}T_{14}\overline{T}_{12}S_{13}. \tag{3.3}$$

Proof. Substituting (3.1) into (1.3), moving all P-elements to the right, one can remove all of them from the both sides of the equality simultaneously. The resulting identity can be rewritten in the form

$$(\overline{T}_{16}^{-1}\overline{T}_{36}^{-1}\overline{T}_{13}\overline{T}_{36})T_{12}\overline{T}_{15}T_{35}\overline{T}_{25} = T_{35}\overline{T}_{25}T_{15}\overline{T}_{13}(T_{24}T_{12}T_{24}^{-1}T_{14}^{-1}).$$

Here nontrivial elements in the subspaces 4 and 6 are contained only in the expressions enclosed in brackets in the r.h.s. and l.h.s., respectively. Consequently, these expressions should be trivial in the subspaces 4 and 6. In this way, we immediately come to the statements of the proposition.

Proposition 5 *Equations (3.2) and (3.3) imply that the elements S and $\overline{S}$ satisfy equations (1.5) –(1.7).*

Proof. Relation (1.5) follows from the identity

$$S_{12}S_{13}S_{23}T_{14}T_{24}T_{34} = S_{23}S_{12}T_{14}T_{24}T_{34},$$

which is proved by successive applications of the first identity from (3.2). Relation (1.6) is proved similarly. As for (1.7), it is a consequence of the identity

$$\overline{S}_{12}S_{13}\overline{S}_{14}S_{24}\overline{S}_{34}T_{15}T_{35}\overline{T}_{16}T_{26}\overline{T}_{36}T_{46}\overline{T}_{56} =$$

$$= S_{24}\overline{S}_{34}S_{14}\overline{S}_{12}S_{13}T_{15}T_{35}\overline{T}_{16}T_{26}\overline{T}_{36}T_{46}\overline{T}_{56},$$

which can also be proved by successive applications of the first identity from (3.2) and (3.3).

The next proposition concerns the similar statement for the FSE.

Proposition 6 *The FSE (1.4) for an element B of the form (1.9) is equivalent to equations (1.5) – (1.7).*

The proof is similar to that of Proposition 4.

Thus, relations (1.5) – (1.7) enable us to construct a special class of solutions for the TE and FSE. Let us turn to particular examples.

Example 1 Take for the algebra A the space of birational transformations $\text{Aut}(\mathbb{C}(x))$ of the field of rational expressions in one indeterminate $\mathbb{C}(x)$, and identify $A \otimes A \otimes \ldots$ with $\text{Aut}(\mathbb{C}(x, y, \ldots))$. Interpreting $\mathbb{C}(x, y, \ldots)$ as the space of rational functions on $\mathbb{C} \times \mathbb{C} \times \ldots$, define

$$(T\varphi)(x, y) = \varphi(xy, y - xy), \quad \overline{T} = T^{-1}.$$

Then, relations (3.2), (3.3) as well as (1.5) – (1.7) are satisfied with

$$(S\varphi)(x, y) = \varphi(xy, [xy]/[x]), \quad [x] = x/(1 - x), \quad \overline{S} = S^{-1}.$$

One can show that the corresponding element R^T is equivalent to the solution Φ_0 for the TE from [13], which in turn is associated with the three-dimensional Hirota equation of the discrete Toda system [11].

Example 2 Let now $\vec{x} = (x_1, x_2)$ be a pair of indeterminates, and put $A = \text{Aut}(\mathbb{C}(\vec{x}))$, $A \otimes A \otimes \ldots$ being identified with $\text{Aut}(\mathbb{C}(\vec{x}, \vec{y}, \ldots))$. Consider the following rational mappings:

$$\vec{x} \cdot \vec{y} = (x_1 y_1, x_1 y_2 + x_2), \quad \vec{x} * \vec{y} = [\vec{x}]^{-1} \cdot [\vec{x} \cdot \vec{y}],$$

where

$$\vec{x}^{-1} = (1/x_1, -x_2/x_1), \quad [\vec{x}] = (x_1/(1 - x_1), x_2/(1 - x_1)),$$

and define

$$(S\varphi)(\vec{x}, \vec{y}) = \varphi(\vec{x} \cdot \vec{y}, \vec{x} * \vec{y}), \quad \overline{S} = S^{-1}.$$

Then, equations (1.5) – (1.7) are satisfied.

Example 3 Let algebra A be the Heisenberg algebra, generated by elements $\{H, \Lambda, 1\}$, satisfying the Heisenberg commutation relation

$$\Lambda H - H\Lambda = 1/h,$$

with h being a complex parameter with a positive real part. Define the function

$$(x; q)_\infty = \prod_{n=0}^{\infty} (1 - xq^n),$$

where $q = \exp(-h)$, and put

$$S = q^{H \otimes \Lambda}(-q^\Lambda \otimes q^{-H}q^{-\Lambda}; q)_\infty^{-1}.$$

Note, that this formula is a specialization of the canonical element in the Heisenberg double of the Borel subalgebra of $U_q(sl(2))$ quantum group, see [12]. Now, both choices of $\overline{S}$, either

$$\overline{S} = S^{-1},$$

or

$$\overline{S} = q^{-H \otimes \Lambda},$$

solve the system (1.5) – (1.7). The corresponding solutions (1.8) and (1.9) to the TE and FSE first have been found in [24].

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4 O-double construction

One particular class of solutions to the system $(1.5) - (1.7)$ is connected with O-doubles [21], which generalize the Heisenberg double of a Hopf algebra [22, 1, 23].

Consider elements S and $\overline{S}$, satisfying (1.5), (1.6) but instead of (1.7), impose more restrictive set of equations:

$$S_{13}\overline{S}_{23} = \overline{S}_{23}S_{13},$$

$$S_{12}\overline{S}_{13}\overline{S}_{23} = \overline{S}_{23}S_{12}, \quad S_{23}S_{13}\overline{S}_{12} = \overline{S}_{12}S_{23}, \tag{4.1}$$

which imply also (1.7). It appears that there is a general algebraic structure, underlying equations (1.5), (1.6), and (4.1).

Let X be a Hopf algebra. In a linear basis $\{e_i\}$ the product, the coproduct, the unit, the counit, and the antipode take the form

$$e_i e_j = m_{ij}^k e_k, \quad \Delta(e_i) = \mu_i^{jk} e_j \otimes e_k,$$

$$1 = \varepsilon^i e_i, \quad \varepsilon(e_i) = \varepsilon_i, \quad \gamma(e_i) = \gamma_i^j e_j, \tag{4.2}$$

where summation over repeated indices is implied. Here m_{ij}^k, μ_i^{jk}, ε^i, ε_i, and γ_i^j are numerical structure constants of the algebra.

Let X^* be the dual Hopf algebra. Following [21], consider an algebra $X^* X X^*$ (O-double), generated by right derivations R_x^*, $x \in X$:

$$R_x^*\colon X^* \to X^*, \quad \langle R_x^*(f), y \rangle = \langle f, R_x(y) \rangle = \langle f, yx \rangle, \tag{4.3}$$

left, L_f, and right, $R_{\gamma^{-1}(g)}$, multiplications , $f, g \in X^*$:

$$L_f, \ R_g\colon X^* \to X^*, \quad L_f(g) = R_g(f) = fg. \tag{4.4}$$

Proposition 7 *The algebra $X^* X X^*$ is an associative algebra, generated by elements $\{e^i, e_j, \tilde{e}^k\}$, and the following defining relations:*

$$e^i e^j = \mu_k^{ij} e^k, \quad e_i e_j = m_{ij}^k e_k, \quad \tilde{e}^i \tilde{e}^j = \mu_k^{ij} \tilde{e}^k,$$

$$e_i e^j = m_{kl}^j \mu_i^{lm} e^k e_m, \quad \tilde{e}^i e_j = \mu_j^{kl} m_{lm}^i e_k \tilde{e}^m, \quad e^i \tilde{e}^j = \tilde{e}^j e^i. \tag{4.5}$$

Proof. One has just to write the compositions of the operations, defined in (4.3) and (4.4), for elements of the linear basis, using (4.2) and the corresponding relations for the dual algebra.

Proposition 8 *Two canonical elements $S = e_i \otimes e^i$, $\overline{S} = e_i \otimes \tilde{e}^i$ in the O-double $X^* X X^*$ satisfy equations (1.5), (1.6), and (4.1).*

The proof is straightforward through substitution of the canonical elements into the relations to be proved, and application of formulae (4.5).

Thus , we have obtained a particular class of general algebraic solutions to the system $(1.5) - (1.7)$, which in turn imply the TE for the element (1.8) and the FSE for the element (1.9).

5 Summary

Solutions for the system of equations (1.5) – (1.7) provide us both with solutions for the three- and four-simplex equations (1.3) and (1.4) through formulae (1.8) and (1.9), respectively.

The O-double of a Hopf algebra provides an algebraic structure, underlying the system (1.5), (1.6) and (4.1), which implies also (1.7). Nevertheless, examples of solutions to the system (1.5) – (1.7), described in Sec. 3, do not come from the O-double construction. This suggests, that the latter is a particular case of a more general algebraic structure, lying behind the system (1.5) – (1.7) itself.

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